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MINIMUM DISTANCE TO THE RANGE OF A BANDED LOWER TRIANGULAR TOEPLITZ OPERATOR IN ℓ^1 AND APPLICATION IN ℓ^1-OPTIMAL CONTROL*

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Abstract. The subject of the paper is the best approximation in ℓ^1 of a given infinite sequence by sequences in the range of a given banded lower-triangular Toeplitz operator. Necessary and sufficient conditions for the existence of a minimizing solution are established and a numerical algorithm for finding such a solution is designed and theoretically founded. It is also shown that an optimal error sequence is only finitely nonzero. The relevancy of the problem in systems theory is outlined and numerical examples are presented.

Key words. Toeplitz operator, banded, lower-triangular, range, minimum distance problem, ℓ^1-optimal control

AMS subject classifications. 47B35, 90C46, 90C90

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1. Introduction. This paper concerns the problem of finding the distance between a given sequence in ℓ^1 and the range of a given infinite lower-triangular Toeplitz band matrix on ℓ^1. Our research was motivated by recent attempts [7], [8], [9], [15], [17], [18] to solve the standard ℓ^1-optimal control problem [24] without using interpolation. The approach of the present paper follows the line of reasoning pursued by Dahleh and Pearson in their seminal paper [10], which launched a hunt for reliable procedures for ℓ^1-optimal control design, but the problem is posed in a different setting, which leads to a different numerical algorithm. Using new theoretical tools, we establish (the already known) necessary and sufficient conditions for the solvability of the problem and design a new numerical algorithm for finding a solution. The convergence of this algorithm is rigorously proved. In essence, our approach shares the spirit of polynomial methods with Casavola's approach [7], [8], [9] in that we formulate the problem using equations with polynomials and power series. In contrast to previous work, which used interpolation techniques (see [11] for a detailed exposition) relying on the solution of numerically ill-conditioned Vandermonde systems, our algorithm is based on finding optimal solutions of overdetermined but numerically better behaved Toeplitz systems. Compared to Casavola's polynomial approach, ours does not introduce any new *structural* parameter (a term coined by Casavola) for the optimization and the Diophantine equation $AX + Y = B$ in the ring of *stable* functions is attacked directly using powerful results from Toeplitz operator theory. The rigorousness of this paper might perhaps seem unnecessary for control of single-input-single-output (SISO) plants, because it is well known that the design of an ℓ^1-optimal controller

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simply consists in designing a dead-beat controller minimizing the sum of the absolute values of the closed-loop impulse response and that hence not much space is left for innovations. But the theoretical rigorousness pays back in that the extension to the multiple-input-multiple-output (MIMO) case is straightforward; preliminary achievements have recently been presented by the authors in [16].

2. Statement of the problem. We start with some standard definitions [4], [5], [6]. The Wiener algebra W is the Banach algebra of all complex-valued functions on the complex unit circle $\mathbf{T}$ with absolutely convergent Fourier series. Thus, a function $a : \mathbf{T} \to \mathbf{C}$ belongs to W if and only if

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j \quad (t = e^{i\theta} \in \mathbf{T}), \quad \|a\|_W := \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

Wiener's theorem states that if $a \in W$ has no zeros on $\mathbf{T}$, then $1/a$ is also in W. For $a \in W$, the infinite Toeplitz matrix $T(a)$ and the finite Toeplitz matrices $T_k(a)$ are defined by $T(a) = (a_{i-j})_{i,j=1}^{\infty}$ and $T_k(a) = (a_{i-j})_{i,j=1}^{k}$, respectively.

We denote by c_0 the set of all real-valued sequences $x = \{x_j\}_{j=1}^{\infty}$ with $|x_j| \to 0$ as $j \to \infty$ and by ℓ^1 the set of all real-valued sequences $x = \{x_j\}_{j=1}^{\infty}$ satisfying $\sum_{j=1}^{\infty} |x_j| < \infty$. The sets c_0 and ℓ^1 are real Banach spaces under the norms $\|x\|_\infty = \sup_{j\geq 1} |x_j|$ and $\|x\|_1 = \sum_{j=1}^{\infty} |x_j|$, respectively. Moreover, ℓ^1 is the dual space of c_0, $\ell^1 = c_0^*$, under the pairing $\langle z, b\rangle = \sum_{j=1}^{\infty} z_j b_j$, $\{z_j\} \in c_0$, $\{b_j\} \in \ell^1$.

Given a Banach space X, we let $\mathcal{B}(X)$ stand for the Banach algebra of all bounded linear operators on X. For $A \in \mathcal{B}(X)$, the norm $\|A\|_{\mathcal{B}(X)}$ is $\sup \|Ax\|$, the supremum over all $x \in X$ with $\|x\| \leq 1$, and the range and null space of A are defined by $\mathcal{R}(A) = A(X)$ and $N(A) = \{x \in X : Ax = 0\}$.

Let $a \in W$ and suppose the Fourier coefficients of a are all real. Then the infinite Toeplitz matrix $T(a)$ induces a bounded linear operator on c_0 and ℓ^1 via

$$(T(a)x)_i = \sum_{j=1}^{\infty} a_{i-j} x_j \quad (j \geq 1).$$

For $j \in \mathbf{Z}$, define the function χ_j by $\chi_j(t) = t^j$ $(t \in \mathbf{T})$. The operators $T(\chi_1)$ and $T(\chi_{-1})$ are the forward and backward shifts acting by the rules

$$T(\chi_1) : \{x_1, x_2, x_3, \ldots\} \mapsto \{0, x_1, x_2, \ldots\},$$
$$T(\chi_{-1}) : \{x_1, x_2, x_3, \ldots\} \mapsto \{x_2, x_3, x_4, \ldots\}.$$

Clearly, $T(a) = \sum_{j=-\infty}^{\infty} a_j T(\chi_j)$. This implies that $\|T(a)\|_{\mathcal{B}(c_0)} = \|T(a)\|_{\mathcal{B}(\ell^1)} = \|a\|_W$. The adjoint operator of $T(a) : c_0 \to c_0$ is the operator $T(\overline{a}) : \ell^1 \to \ell^1$, where $\overline{a}(t) = \sum_{j=-\infty}^{\infty} a_j t^{-j}$ $(t = e^{i\theta} \in \mathbf{T})$.

Throughout this paper we suppose that $a_+(t) = a_0 + a_1 t + \cdots + a_n t^n$ with real numbers $a_0, a_1, \ldots, a_n$ and with $a_n \neq 0$. Clearly, $T(a_+)$ is a banded lower-triangular Toeplitz matrix, while $T(\overline{a}_+)$ is a banded upper-triangular Toeplitz matrix. We always think of $T(a_+)$ as acting on ℓ^1 and always consider $T(\overline{a}_+)$ as an operator on c_0. Thus, $T(a_+)$ is the adjoint of $T(\overline{a}_+)$.

This paper concerns the following problem of finding the distance between some given real sequence with finite sum of absolute values and the range of a banded lower triangular Toeplitz operator in ℓ^1 space. Given $b \in \ell^1$, determine the distance

$$d := \operatorname{dist}_{\ell^1}(b, \mathcal{R}(T(a_+))) := \inf_{m \in \mathcal{R}(T(a_+))} \|b - m\|_1,$$

find out whether there is an $m_0 \in \mathcal{R}(T(a_+))$ with $\|b - m_0\|_1 = d$, and if yes, compute such an m_0. Note that once m_0 is available, we can easily solve the lower-triangular system $T(a_+)x_0 = m_0$ to get x_0.

3. Two results from functional analysis. We will employ the following two theorems (whose proofs can be found in [21, pp. 121, 156]). Recall that the annihilator $M^\perp$ of a set $M \subset X$ is defined as $M^\perp = \{b \in X^* : \langle z, b\rangle = 0 \text{ for all } z \text{ in } M\}$. Furthermore, two elements $z \in X$ and $b \in X^*$ are said to be aligned if the equality $\|z\|\,\|b\| = \langle z, b\rangle$ holds.

THEOREM 3.1. *Let M be a linear subset of a real normed space X and let $b \in X^*$. Then*

$$\inf_{m \in M^\perp} \|b - m\| = \sup_{z \in M, \|z\| \le 1} \langle z, b\rangle. \tag{3.1}$$

The infimum in (3.1) *is always attained at some $m_0 \in M^\perp$. If the supremum in* (3.1) *is achieved for some $z_0 \in M$ with $\|z_0\| \le 1$, then z_0 and $b - m_0$ are aligned.*

THEOREM 3.2. *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed, in which case $\mathcal{R}(A^*) = [N(A)]^\perp$.*

4. Toeplitz operators. The product of two Toeplitz operators is in general not a Toeplitz operator. However, this happens in certain special cases. Let W_+ and W_- denote the functions in W whose Fourier coefficients with negative and positive indices vanish, respectively. Thus, if $c_\pm \in W_\pm$, then $T(c_-)$ and $T(c_+)$ are upper and lower triangular, respectively. It is easily seen by direct inspection that if $c_- \in W_-$, $f \in W$, $c_+ \in W_+$, then

$$T(c_-)T(f)T(c_+) = T(c_- f c_+). \tag{4.1}$$

The following results are known to specialists (see, e.g., [4] and [12]). We include the proofs for the reader's convenience.

PROPOSITION 4.1. *The range $\mathcal{R}(T(a_+))$ is a closed subset of ℓ^1 if and only if a_+ has no zeros on* $\mathbf{T}$.

Proof. If a_+ has no zeros on $\mathbf{T}$, then a_+^{-1} belongs to W and has real Fourier coefficients. From (4.1) we obtain that $T(a_+^{-1})T(a_+) = I$. Thus, $T(a_+)$ has a bounded left inverse, which implies that the range of $T(a_+)$ is closed (see, e.g., [12, section I.1.2]).

Now suppose $a_+(\tau) = 0$ for some $\tau \in \mathbf{T}$. Contrary to what we want, we assume that $\mathcal{R}(T(a_+))$ is a closed subset of ℓ^1. We denote by $\ell^1(\mathbf{C})$ the complex Banach space of all complex-valued sequences $x = \{x_j\}_{j=1}^\infty$ for which $\|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty$. The range of $T(a_+)$ on $\ell^1(\mathbf{C})$ is $\mathcal{R}(T(a_+)) + i\mathcal{R}(T(a_+))$, which is closed whenever $\mathcal{R}(T(a_+))$ is closed. From Theorem 3.2 we now infer that $T(\overline{a}_+) : c_0(\mathbf{C}) \to c_0(\mathbf{C})$ has closed range, where $c_0(\mathbf{C})$ is defined in analogy to $\ell^1(\mathbf{C})$. The operator $T(\overline{a}_+)$ is upper-triangular, and it is easily seen that the range of every nonzero upper-triangular Toeplitz operator contains all finitely supported sequences. Consequently, $T(\overline{a}_+)$ must be surjective. We may write

$$\begin{aligned}\overline{a}_+(t) &= a(1/t) = a_0 + a_1 \frac{1}{t} + \cdots + a_n \frac{1}{t^n} \\ &= a_n\left(\frac{1}{t} - \tau\right)\left(\frac{1}{t} - z_1\right)\ldots\left(\frac{1}{t} - z_{n-1}\right) = a_n(\chi_{-1}(t) - \tau)d(t).\end{aligned}$$

Since $T(\overline{a}_+) = T(\chi_{-1} - \tau)T(d)$ by (4.1), the operator $T(\chi_{-1} - \tau)$ is surjective together with $T(\overline{a}_+)$. The equation $T(\chi_{-1} - \tau)z = 0$ is satisfied if and only if $z_j = \tau^{j-1}z_1$ $(j \geq 1)$, and this is a sequence in $c_0(\mathbf{C})$ only for $z_1 = 0$. Thus, $T(\overline{a}_+)$ is injective on $c_0(\mathbf{C})$. In summary, $T(\chi_{-1} - \tau)$ is invertible on $c_0(\mathbf{C})$. It follows that $T(\chi_1 - 1/\tau)$ is invertible on $\ell^1(\mathbf{C})$. But the solution of $T(\chi_1 - 1/\tau)x = \{1, 0, 0, \ldots\}$ is $x_j = -\tau^j$ $(j \geq 1)$, which is not in $\ell^1(\mathbf{C})$. This contradiction proves that $\mathcal{R}(T(a_+))$ cannot be closed. □

The function $a_+(z) = a_0 + a_1 z + \cdots + a_n z^n$ is defined for all $z \in \mathbf{C}$.

PROPOSITION 4.2. *If a_+ has no zeros on $\mathbf{T}$, then the dimension of $N(T(\overline{a}_+))$ in c_0 is equal to the number of zeros of a_+ in the open unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$.*

Proof. Let a_+ have $\varkappa$ zeros $\delta_1, \ldots, \delta_\varkappa$ in $\mathbf{D}$ and $n - \varkappa$ zeros $\mu_1, \ldots, \mu_{n-\varkappa}$ in $\mathbf{C} \setminus (\mathbf{D} \cup \mathbf{T})$. We then have

$$\begin{aligned}\overline{a}_+(t) = a_+(1/t) &= a_n \prod_{k=1}^{n-\varkappa} \left(\frac{1}{t} - \mu_k\right) \prod_{j=1}^{\varkappa} \left(\frac{1}{t} - \delta_j\right)\\ &= \gamma t^{-\varkappa} \prod_{k=1}^{n-\varkappa} \left(1 - \frac{1}{\mu_k t}\right) \prod_{j=1}^{\varkappa} (1 - \delta_j t)\end{aligned}$$

with $\gamma = a_n(-\mu_1)\ldots(-\mu_{n-\varkappa})$. We consider $T(\overline{a}_+)$ on $c_0(\mathbf{C})$. Let N be the null space of $T(\overline{a}_+)$ on c_0. Then $N + iN$ is the null space of $T(\overline{a}_+)$ on $c_0(\mathbf{C})$. From (4.1) we obtain that

$$T(\overline{a}_+) = \gamma T(\chi_{-\varkappa}) \prod_{k=1}^{n-\varkappa} \left(I - \frac{1}{\mu_k} T(\chi_{-1})\right) \prod_{j=1}^{\varkappa} (I - \delta_j T(\chi_1)).$$

Since $\|(1/\mu_k)T(\chi_{-1})\| = 1/|\mu_k| < 1$ and $\|\delta_j T(\chi_1)\| = |\delta_j| < 1$, we conclude that the operators $I - (1/\mu_k)T(\chi_{-1})$ and $I - \delta_j T(\chi_1)$ are all invertible. Consequently, the dimension of $N + iN$ is the dimension of the null space of $T(\chi_{-\varkappa})$ on $c_0(\mathbf{C})$. It follows that the dimension of $N + iN$ over $\mathbf{C}$ is $\varkappa$, which implies that the dimension of N over $\mathbf{R}$ is also $\varkappa$. □

5. Existence of the solution. Here is our result on the solvability of the problem posed in section 2. This result states that for some sequence b in ℓ^1 and a polynomial a_+, there is a real sequence in the range of the Toeplitz operator generated by a_+ that minimizes distance to b if and only if the polynomial has no zeros on the unit circle. The error sequence then has a finite number of nonzero terms only.

THEOREM 5.1. *The problem*

$$\|b - m\|_1 = \mathrm{dist}_{\ell^1}(b, \mathcal{R}(T(a_+))) =: d \tag{5.1}$$

has a solution $m_0 \in \mathcal{R}(T(a_+))$ for every $b \in \ell^1$ if and only if a_+ has no zeros on $\mathbf{T}$. If $a_+(t) \neq 0$ for $t \in \mathbf{T}$, then for every $b \in \ell^1$ there exists a $z_0 \in N(T(\overline{a}_+))$ such that

$$\|z_0\|_\infty \leq 1 \quad \text{and} \quad d = \langle z_0, b\rangle = \sup_{z \in N(T(\overline{a}_+)), \|z\|_\infty \leq 1} \langle z, b\rangle, \tag{5.2}$$

and if $m_0 \in \mathcal{R}(T(a_+))$ is any sequence satisfying (5.1), then the sequence $b - m_0$ has only finitely many nonzero terms.

Proof. If a_+ has a zero on $\mathbf{T}$, then $\mathcal{R}(T(a_+))$ is not closed due to Proposition 4.1 and hence (5.1) has no solution $m_0 \in \mathcal{R}(T(a_+))$ if b is in the closure of $\mathcal{R}(T(a_+))$ but not in $\mathcal{R}(T(a_+))$.

Now suppose that a_+ has no zeros on $\mathbf{T}$. Then $\mathcal{R}(T(a_+))$ is closed by Proposition 4.1. From Theorem 3.2 we deduce that $\mathcal{R}(T(a_+)) = [N(T(\overline{a}_+))]^\perp$. The existence of an $m_0 \in \mathcal{R}(T(a_+))$ satisfying (5.1) then follows from Theorem 3.1. This theorem also yields the equality

$$d = \sup_{z \in N(T(\overline{a}_+)), \|z\|_\infty \le 1} \langle z, b\rangle,$$

and since $\{z \in N(T(\overline{a}_+)) : \|z\|_\infty \le 1\}$ is compact by virtue of Proposition 4.2 and the map $z \mapsto \langle z, b\rangle$ is continuous, we conclude that the supremum is attained at some $z_0 \in N(T(\overline{a}_+))$ with $\|z_0\|_\infty \le 1$.

The last assertion of the theorem is trivial for $d = 0$. So let $d > 0$, which implies that $\|z_0\|_\infty > 0$. The sequences $b - m_0$ and z_0 are aligned by Theorem 3.1. Consequently, with $b - m_0 = \{e_j\}_{j=1}^\infty$ and $z_0 = \{z_j\}_{j=1}^\infty$,

$$\sum_{j=1}^\infty z_j e_j = \|z_0\|_\infty \sum_{j=1}^\infty |e_j|. \tag{5.3}$$

As $\{z_j\} \in c_0$, there is a j_0 such that $|z_j| < \|z_0\|_\infty$ for all $j \ge j_0$. From (5.3) we infer that $e_j = 0$ for $j \ge j_0$. □

6. Finite sections of Toeplitz operators. In this section, we quote two known theorems that will be needed when proving the convergence of and giving an error estimate for our numerical algorithm. For $k \ge 1$, we denote by P_k the projection on ℓ^1 and c_0 that acts by the rule

$$P_k : \{x_1, x_2, x_3, \ldots\} \mapsto \{x_1, \ldots, x_k, 0, 0, \ldots\}.$$

We identify $\mathcal{R}(P_k)$ and $\mathbf{R}^k$, and hence we may think of vectors in $\mathbf{R}^k$ as elements of ℓ^1 or c_0. The following theorem was established by Reich [23] and Baxter [1]. Full proofs are also in [4, section 3.3] and [12, section II.2].

THEOREM 6.1. *If $f \in W$ and $T(f)$ is invertible on ℓ^1, then the matrices $T_k(f)$ are invertible for all sufficiently large k and $T_k^{-1}(f)P_k y$ converges in ℓ^1 to $T^{-1}(f)y$ for every $y \in \ell^1$.*

The next theorem can be proved using the asymptotic inverses presented in [4, section 3.5] or [6, section 2.3].

THEOREM 6.2. *Let f be a Laurent polynomial, that is, suppose f has only finitely many nonzero Fourier coefficients, and let $T(f)$ be invertible on ℓ^1. Fix a natural number $\varkappa$. Then there exist a natural number k_0 and constants $\alpha > 0$ and $C < \infty$ such that*

$$\|P_\varkappa T_k^{-1}(f) - P_\varkappa T^{-1}(f)\|_{\mathcal{B}(\ell^1)} = \|T_k^{-1}(\overline{f})P_\varkappa - T^{-1}(\overline{f})P_\varkappa\|_{\mathcal{B}(c_0)} \le C\, e^{-\alpha k}$$

for all $k \ge k_0$.

7. Numerical algorithm. Fix $b \in \ell^1$ and a_+ as above. Suppose a_+ has exactly $\varkappa$ zeros in $\mathbf{D}$ and no zeros on $\mathbf{T}$. If $k \geq \varkappa + 1$, the operator $P_k T(a_+) P_{k-\varkappa}$ may be identified with a $k \times (k-\varkappa)$ matrix. The system $P_k T(a_+) P_{k-\varkappa} x^{(k)} = P_k b$ is overdetermined for $\varkappa \geq 1$. However, we can find an $x_0^{(k)} \in \mathbf{R}^{k-\varkappa}$ such that the residue

$$\|P_k T(a_+) P_{k-\varkappa} x^{(k)} - P_k b\|_1 \tag{7.1}$$

assumes its minimum at $x^{(k)} = x_0^{(k)}$. Let $d = \mathrm{dist}_{\ell^1}(b, \mathcal{R}(T(a_+)))$ and let d_k be the minimal value of (7.1). The following theorem reveals that d_k converges to d exponentially fast.

THEOREM 7.1. *There are constants $E < \infty$ and $\beta > 0$ such that $|d_k - d| \leq E\, e^{-\beta k}$ for all $k \geq 1$.*

Proof. Put $f(t) = t^{-\varkappa} a_+(t) = t^{-\varkappa}(a_0 + a_1 t + \cdots + a_n t^n)$. We claim that $T(f)$ is invertible on ℓ^1. Indeed, the proof of Proposition 4.2 shows that

$$T(\overline{f}) = \gamma \prod_{k=1}^{n-\varkappa} \left(I - \frac{1}{\mu_k} T(\chi_{-1}) \right) \prod_{j=1}^{\varkappa} (I - \delta_j T(\chi_1))$$

with all operators on the right being invertible on $c_0(\mathbf{C})$. It follows that $T(\overline{f})$ is invertible on c_0 and hence that $T(f)$ is invertible on ℓ^1.

From (4.1) we deduce that $T(a_+) = T(f) T(\chi_\varkappa)$. Let $x^{(k)} = \{x_1^{(k)}, \ldots, x_{k-\varkappa}^{(k)}, 0, \ldots\}$ and define $w^{(k)} \in \mathcal{R}(P_k)$ by

$$w^{(k)} = \{\underbrace{0, \ldots, 0}_{\varkappa}, x_1^{(k)}, \ldots, x_{k-\varkappa}^{(k)}, 0, \ldots\}.$$

Let Q_k be given on ℓ^1 and c_0 by $Q_k = I - P_k$, that is,

$$Q_k : \{x_1, x_2, x_3, \ldots\} \mapsto \{0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots\}.$$

We have $P_{k-\varkappa} x^{(k)} = T(\chi_{-\varkappa}) P_k w^{(k)}$, and since $T(\chi_{-\varkappa}) T(\chi_\varkappa) = Q_k$ and $Q_\varkappa P_k = P_k Q_\varkappa$, we get

$$\begin{aligned} &\|P_k b - P_k T(a_+) P_{k-\varkappa} x^{(k)}\|_1 = \|P_k b - P_k T(f) T(\chi_\varkappa) T(\chi_{-\varkappa}) P_k w^{(k)}\|_1 \\ &= \|P_k b - P_k T(f) P_k Q_\varkappa w^{(k)}\|_1 = \|P_k b - T_k(f) Q_\varkappa w^{(k)}\|_1. \end{aligned} \tag{7.2}$$

The minimum of (7.2) as $w^{(k)}$ ranges over $\mathbf{R}^k$ is d_k, and the minimum is attained at the $w_0^{(k)}$ corresponding to any $x_0^{(k)}$ that minimizes (7.1). Hence, by Theorems 3.1 and 3.2,

$$d_k = \sup_{z \in N(Q_\varkappa T_k(\overline{f})), \|z\|_\infty \leq 1} \langle z, P_k b \rangle. \tag{7.3}$$

Theorem 6.1 implies that there is a k_0 such that the matrices $T_k(f)$ are invertible for all $k \geq k_0$. Let $k \geq k_0$. We have $Q_\varkappa T_k(\overline{f}) z = 0$ if and only if there is a $y \in \mathbf{R}^\varkappa$ such that $T_k(\overline{f}) z = P_\varkappa y$ or, equivalently, $z = T_k^{-1}(\overline{f}) P_\varkappa y$. (Note that $T_k(\overline{f})$ is simply the transpose of $T_k(f)$). From (7.3) we therefore obtain

$$\begin{aligned} d_k &= \sup_{z = T_k^{-1}(\overline{f}) P_\varkappa y, \|z\|_\infty \leq 1} \langle z, P_k b \rangle \\ &= \sup_{\|T_k^{-1}(\overline{f}) P_\varkappa y\|_\infty \leq 1} \langle T_k^{-1}(\overline{f}) P_\varkappa y, P_k b \rangle \\ &= \sup_{\|T_k^{-1}(\overline{f}) P_\varkappa y\|_\infty \leq 1} \langle P_\varkappa y, P_\varkappa T_k^{-1}(f) P_k b \rangle. \end{aligned}$$

Put $\mathcal{M}_k = \{y \in \mathbf{R}^\varkappa : \|T_k^{-1}(\overline{f})P_\varkappa y\|_\infty \le 1\}$ and define $\varphi_k : \mathcal{M}_k \to \mathbf{R}$ by $\varphi_k(y) = \langle y, P_\varkappa T_k^{-1}(f)P_k b\rangle$. Then

$$d_k = \sup_{y\in\mathcal{M}_k} \varphi_k(y).$$

From Theorem 5.1 we know that

$$d = \sup_{z\in N(T(\overline{a}_+)),\|z\|_\infty\le 1} \langle z, b\rangle.$$

As $T(\overline{a}_+) = T(\chi_{-\varkappa})T(\overline{f})$, the equation $T(\overline{a}_+)z = 0$ is equivalent to the equation $T(\chi_{-\varkappa})T(\overline{f})f = 0$, that is, to the existence of a $y \in \mathbf{R}^\varkappa$ such that $z = T^{-1}(\overline{f})P_\varkappa y$. It follows that

$$\begin{aligned} d &= \sup_{z=T^{-1}(\overline{f})P_\varkappa y,\|z\|_\infty\le 1} \langle z, b\rangle \\ &= \sup_{\|T^{-1}(\overline{f})P_\varkappa y\|_\infty\le 1} \langle T^{-1}(\overline{f})P_\varkappa y, b\rangle \\ &= \sup_{\|T^{-1}(\overline{f})P_\varkappa y\|_\infty\le 1} \langle P_\varkappa y, P_\varkappa T^{-1}(f)b\rangle = \sup_{y\in\mathcal{M}} \varphi(y), \end{aligned}$$

where $\mathcal{M} = \{y \in \mathbf{R}^\varkappa : \|T^{-1}(\overline{f})P_\varkappa y\|_\infty \le 1\}$ and $\varphi : \mathcal{M} \to \mathbf{R}$ is given by $\varphi(y) = \langle y, P_\varkappa T^{-1}(f)b\rangle$. By Theorem 6.2,

$$\varphi_k(y) = \sum_{j=1}^{\varkappa} \gamma_j(k)y_j, \quad \varphi(y) = \sum_{j=1}^{\varkappa} \gamma_j y_j,$$

where $\gamma_j(k)$ converges to γ_j exponentially fast as $k \to \infty$. We remark that if $y \in \mathcal{M}_k$, then

$$\begin{aligned} \|P_\varkappa y\|_\infty &\le \|P_k T(\overline{f})P_k\|_{\mathcal{B}(c_0)}\|T_k^{-1}(\overline{f})P_\varkappa y\|_\infty \\ &\le \|f\|_W \|T_k^{-1}(\overline{f})P_\varkappa y\|_\infty \le \|f\|_W. \end{aligned}$$

Analogously, $\|P_\varkappa y\|_\infty \le \|f\|_W$ for $y \in \mathcal{M}$.

Now take $y_0 = (y_1^{(0)}, \dots, y_\varkappa^{(0)}) \in \mathcal{M}$ so that $\varphi(y_0) = d$. Theorem 6.2 yields

$$\begin{aligned} \|T_k^{-1}(\overline{f})P_\varkappa y_0\|_\infty &\le \|T^{-1}(\overline{f})P_\varkappa y_0\|_\infty + \|T_k^{-1}(\overline{f})P_\varkappa y_0 - T^{-1}(\overline{f})P_\varkappa y_0\|_\infty \\ &\le 1 + C\,e^{-\alpha k}\|P_\varkappa y_0\|_\infty \le 1 + C\,e^{-\alpha k}\|f\|_W =: 1 + \sigma_k. \end{aligned}$$

Thus, $(1 + \sigma_k)^{-1}y_0 \in \mathcal{M}_k$. This implies that

$$d_k \ge \varphi_k[(1 + \sigma_k)^{-1}y_0] = (1 + \sigma_k)^{-1} \sum_{j=1}^{\varkappa} \gamma_j(k)y_j^{(0)}.$$

Since $\{\gamma_j(k) - \gamma_j\}_{k=1}^\infty$ is exponentially decaying for each j, we have

$$\sum_{j=1}^{\varkappa} \gamma_j(k)y_j^{(0)} \ge \sum_{j=1}^{\varkappa} \gamma_j y_j^{(0)} - \tau_k = d - \tau_k$$

with some exponentially decaying sequence $\{\tau_k\}$. In summary, we have shown that $(1+\sigma_k)d_k \geq d - \tau_k$, which gives

$$d - d_k \leq \sigma_k d_k + \tau_k \leq \sigma_k \|b\|_1 + \tau_k. \tag{7.4}$$

Again taking into account that $\{\gamma_j(k) - \gamma_j\}_{k=1}^{\infty}$ is exponentially decaying for each j and using that $\|P_\varkappa y\|_\infty \leq \|f\|_W$ for all $y \in \mathcal{M}_k$, we obtain

$$d_k = \sup_{y\in\mathcal{M}_k} \sum_{j=1}^{\varkappa} \gamma_j(k) y_j \leq \sup_{y\in\mathcal{M}_k} \sum_{j=1}^{\varkappa} \gamma_j y_j + \varrho_k$$

with an exponentially decaying sequence $\{\varrho_k\}$. For $y \in \mathcal{M}_k$, Theorem 6.2 gives

$$\begin{aligned}\|T^{-1}(\overline{f})P_\varkappa y\|_\infty &\leq \|T_k^{-1}(\overline{f})P_\varkappa y\|_\infty + \|T^{-1}(\overline{f})P_\varkappa y - T_k^{-1}(\overline{f})P_\varkappa y\|_\infty \\ &\leq 1 + C\, e^{-\alpha k}\|P_\varkappa y\|_\infty \leq 1 + C\, e^{-\alpha k}\|f\|_W =: 1+\sigma_k,\end{aligned}$$

and therefore $(1+\sigma_k)^{-1}y \in \mathcal{M}$. It follows that

$$\begin{aligned}d_k &\leq \sup_{(1+\sigma_k)^{-1}y\in\mathcal{M}} \sum_{j=1}^{\varkappa} \gamma_j y_j + \varrho_k = \sup_{v\in\mathcal{M}} \sum_{j=1}^{\varkappa} \gamma_j \cdot (1+\sigma_k)v_j + \varrho_k \\ &= (1+\sigma_k) \sup_{v\in\mathcal{M}} \varphi(v) + \varrho_k \leq (1+\sigma_k)d + \varrho_k,\end{aligned}$$

whence

$$d_k - d \leq \sigma_k d + \varrho_k. \tag{7.5}$$

Combining (7.4) and (7.5) we arrive at the assertion. □

Corollary 7.2. *For each $k \geq 1$, let $x_0^{(k)} \in \mathcal{R}(P_{k-\varkappa})$ be an element at which (7.1) attains its minimum d_k. If $k_i \to \infty$ and $\{x_0^{(k_i)}\}_{i=1}^{\infty}$ is any sequence that converges in ℓ^1 to some $x_0 \in \ell^1$, then $\|b - T(a_+)x_0\|_1 = d$.*

Proof. If $\|P_{k_i}b - P_{k_i}T(a_+)P_{k_i-\varkappa}x_0^{(k_i)}\|_1 = d_{k_i}$ and $x_0^{(k_i)} \to x$ as $i \to \infty$, then $\|b - T(a_+)x_0\|_1 = d$ because $d_{k_i} \to d$ by Theorem 7.1. □

8. Error estimate. In practice, we have an $x_0^{(k)}$ with

$$\|P_k b - P_k T(a_+)P_{k-\varkappa}x_0^{(k)}\|_1 = d_k,$$

and $\widetilde{m}_0 = T(a_+)P_{k-\varkappa}x_0^{(k)}$ is taken as an approximate solution. The question is, how far is $\|b - \widetilde{m}_0\|_1$ away from the optimal value d? We have

$$\begin{aligned}\|b - \widetilde{m}_0\|_1 \leq{}& \|b - P_k b\|_1 + \|P_k b - P_k T(a_+)P_{k-\varkappa}x_0^{(k)}\|_1 \\ &+ \|P_k T(a_+)P_{k-\varkappa}x_0^{(k)} - T(a_+)P_{k-\varkappa}x_0^{(k)}\|_1.\end{aligned} \tag{8.1}$$

Clearly, $\|b - P_k b\|_1 = \|Q_k b\|_1 = o(1)$ is given a priori. The second term on the right of (8.1) is just d_k. Let $x_i^{(k)}$ $(i = 1, \ldots, k-\varkappa)$ denote the components of $x_0^{(k)}$. The vector in the third term on the right of (8.1) is

$$-Q_k T(a_+) P_{k-\varkappa} x_0^{(k)} = -\begin{pmatrix} a_k & \dots & a_{\varkappa+1} \\ a_{k+1} & \dots & a_{\varkappa+2} \\ \vdots & & \vdots \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_{k-\varkappa}^{(k)} \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_{\varkappa+1} \\ 0 & \dots & 0 & 0 & a_n & \dots & a_{\varkappa+2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & a_n \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_{k-\varkappa}^{(k)} \end{pmatrix},$$

which implies that

$$\|Q_k T(a_+) P_{k-\varkappa} x_0^{(k)}\|_1 \le (|a_{\varkappa+1}| + \dots + |a_n|) \left(|x_{k-n+1}^{(k)}| + \dots + |x_{k-\varkappa}^{(k)}| \right)$$
$$\le \|a_+\|_W \, \|Q_{k-n} x_0^{(k)}\|_1.$$

The vector $x_0^{(k)}$ is available and $\|Q_{k-n} x_0^{(k)}\|_1$ is the ℓ^1 norm of the last $n - \varkappa$ components of $x_0^{(k)}$. If k is large, then the last $n - \varkappa$ components of $x_0^{(k)}$ are expected to be small. In summary, (8.1) and Theorem 7.1 yield

$$\|b - \widetilde{m}_0\|_1 \le \|Q_k b\|_1 + \|a_+\|_W \, \|Q_{k-n} x_0^{(k)}\|_1 + d + \text{ exponentially small term.}$$

If $\varkappa = n$, then $-Q_k T(a_+) P_{k-\varkappa} x_0^{(k)} = 0$, and hence we even have

$$\|b - \widetilde{m}_0\|_1 \le \|Q_k b\|_1 + d + \text{ exponentially small term.}$$

Finally, if $\varkappa = n$ and b is finitely supported, then

$$\|b - \widetilde{m}_0\|_1 \le d + \text{ exponentially small term.}$$

9. Numerical example. The following example illustrates the algorithm described above. We consider the banded lower triangular Toeplitz matrix $T(a_+)$ with the symbol

$$a_+(t) = -0.1224 - 0.2906t + 0.7122t^2 + 2.7983t^3 + 2.9168t^4 + t^5$$

and we are looking for a sequence $x \in \ell^1$ minimizing $\|b - T(a_+)x\|_1$ for the right-hand side

$$b = \{1.8645, -0.3398, -1.1398, -0.2111, 1.1902, -1.1162, 0, 0, \dots\}.$$

The five zeros of the polynomial $a_+(t)$ are all inside the open unit disk. Thus, we can proceed as in section 7 with $\varkappa = 5$. (Notice that the algorithm of section 7 would be applicable to $\varkappa < 5$ as well.) Accordingly, we approximate $T(a_+)$ by the finite matrices $A_k = P_{5+k} T(a_+) P_k$ $(k = 1, 2, \dots)$. For example,

$$A_3 = \begin{pmatrix} -0.1224 & 0 & 0 \\ -0.2906 & -0.1224 & 0 \\ 0.7122 & -0.2906 & -0.1224 \\ 2.7983 & 0.7122 & -0.2906 \\ 2.9168 & 2.7983 & 0.7122 \\ 1 & 2.9168 & 2.7983 \\ 0 & 1 & 2.9168 \\ 0 & 0 & 1 \end{pmatrix}.$$

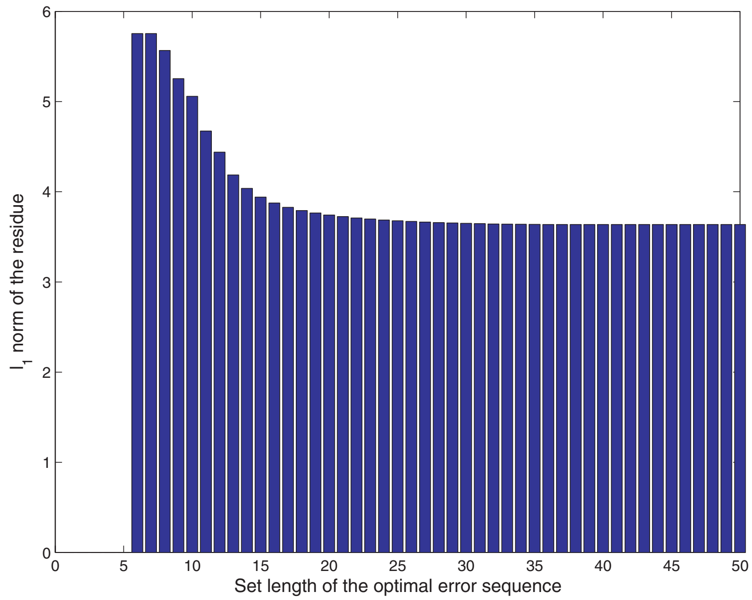

FIG. 9.1. *Evolvement of the ℓ^1-norm of the residue with increasing set length of the optimal error sequence.*

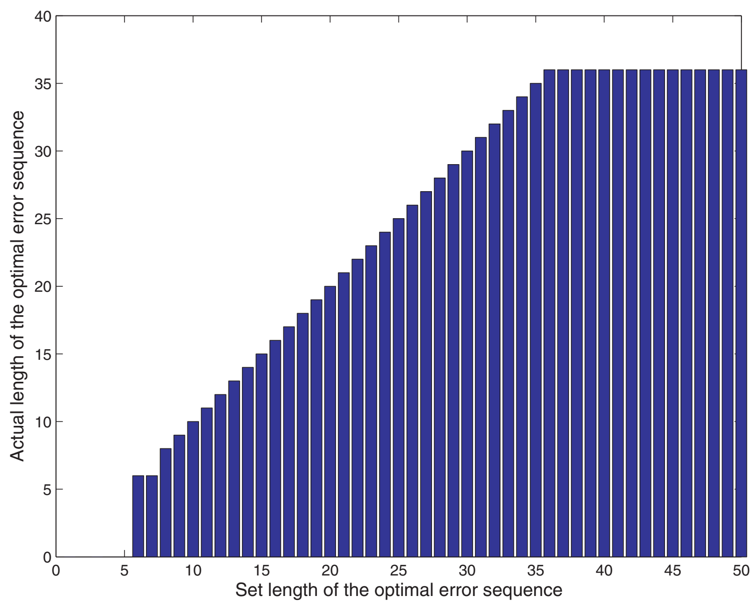

FIG. 9.2. *Evolvement of the actual length of the optimal error sequence with increasing set length of the optimal error sequence.*

Solving the corresponding overdetermined linear system for a solution minimizing the ℓ_1-norm of the residue (using a general LP solver) and repeating this for increasing index k, we obtain Figures 9.1 and 9.2. In Figure 9.1 we nicely see the exponentially fast stabilization of the objective function (that is, the number d_k or, equivalently, the ℓ^1-norm of the residuum) predicted by Theorem 7.1. Figure 9.2 reveals that for

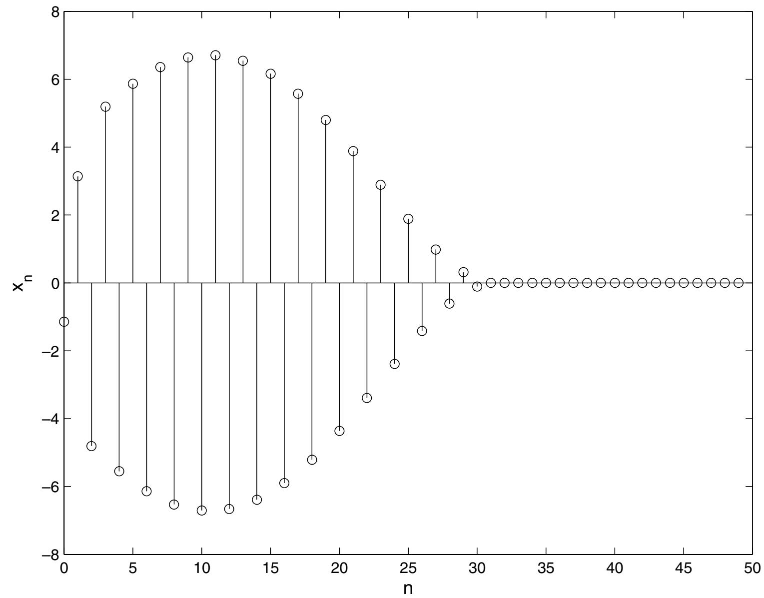

FIG. 9.3. *A sequence $x \in \ell_1$ minimizing $\|b - T(a_+)x\|_1$.*

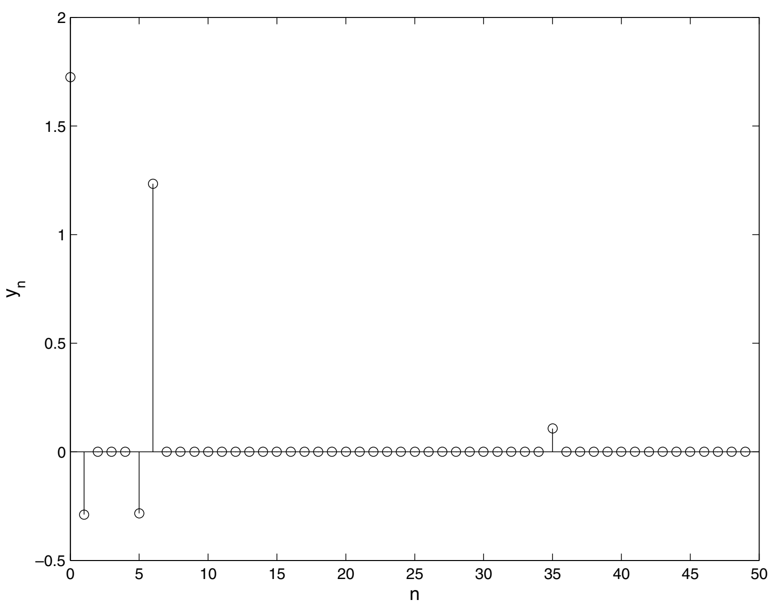

FIG. 9.4. *The optimal error sequence (residue) $y = b - T(a_+)x$.*

$k - \varkappa \geq 36$ the number of nonzero terms in $P_{k-\varkappa}x_0^{(k)}$ is no longer increasing. Thus, although we offer more and more space to the approximate optimal solution sequence $P_{k-\varkappa}x_0^{(k)}$, its actual length settles at 36.

Figures 9.3 and 9.4 show an optimal solution x and the residue sequence $y = b - T(a_+)x$. The very small number of nonzero terms in Figure 9.4 is a mystery we cannot yet explain.

10. Improvement of conditioning. Both our Toeplitz approach and the Vandermonde interpolation approach of [10] lead to linear systems of equations. These can be tackled by invoking an LP solver. Numerical experiments show that the condition numbers of the matrices emerging in our algorithm are much smaller than those of the matrices that result from interpolation. To give a concrete example, consider the task of finding the distance between a given $b \in \ell^1$ and the range of the Toeplitz operator $T(a_+)$ with $a_+(t)$ having its 10 roots equally distributed in the interval $[0.5, 0.9]$, that is, $a_+(t) = 0.0238 - 0.3520t + 2.3334t^2 - 9.1302t^3 + 23.3525t^4 - 40.7975t^5 + 49.3052t^6 - 40.7037t^7 + 21.9685t^8 - 7t^9 + t^{10}$. Let us set the length of the approximate optimal error sequence to 13. The 2-norm condition number, that is, the ratio of the largest and the smallest singular value, of the 10×13 Vandermonde matrix V_{13} built from the roots of $a_+(t)$ equals $\kappa(V_{13}) = 9.5458 \cdot 10^9$. In contrast to this, the 2-norm condition number of the matrix $A_{13} = P_{13}T(a_+)P_3$ is $\kappa(A_{13}) = 14.948$. To make a general conclusion, Vandermonde systems are known to be extremely ill-conditioned unless the roots are distributed along the unit circle. Analytical expressions for the conditioning of some common distributions are in [13, p. 418]. Similar analytical results for Toeplitz systems are not known to the authors.

11. Application to ℓ^1-optimal control. In compliance with control engineering, we now use the variable λ instead of t. (The variables λ, d, and z^{-1} are more common in control theory literature than t. The variable t is dominating in Toeplitz operators theory but it could be confused with time.) It is well known [19] that the achievable internally stable closed-loop maps $y(\lambda)$ of a standard feedback connection are parametrized by $y(\lambda) = b(\lambda) - a(\lambda)x(\lambda)$, where $a(\lambda)$, $b(\lambda)$, $x(\lambda)$, $y(\lambda)$ are power series with coefficient sequences in ℓ^1, $a(\lambda)$ and $b(\lambda)$ are given, and $x(\lambda)$, which is also called the Youla–Kučera parameter, is unknown. For finite-dimensional systems, $a(\lambda)$ is actually a polynomial. A principal task of ℓ^1-control is to design a stabilizing controller that minimizes the ℓ^1-norm of the coefficient sequence of $y(\lambda)$. This controller will guarantee optimal attenuation of peaks in the error signal $y(\lambda)$ because the Wiener norm of $y(\lambda)$ (= ℓ^1-norm of its coefficient sequence) is equal to the ℓ^∞-induced operator norm of the closed-loop system [24].

Any stabilizing controller is determined by its Youla–Kučera parameter $x(\lambda)$. As the coefficient sequence of $a(\lambda)x(\lambda)$ results from that of $x(\lambda)$ by the action of an infinite lower-triangular Toeplitz band matrix, the problem of designing an optimal controller leads to the minimum distance problem between a given sequence $b \in \ell^1$ and the range $\mathcal{R}(T(a))$ of the infinite lower-triangular Toeplitz matrix $T(a)$ in ℓ^1. A concrete design procedure will be exemplified in the next section.

A noteworthy feature of the approach proposed here is that the Youla–Kučera parameter is explicitly computed and that, consequently, there is no need to extract a controller from the optimal closed-loop transfer function.

12. A concrete ℓ^1-optimal control design. We consider the standard feedback configuration of Figure 12.1 with a discrete-time plant $G(\lambda)$. Our aim is to construct a stabilizing discrete-time controller $C(\lambda)$ that minimizes the Wiener norm of the sensitivity function of the closed-loop system, that is, of the transfer function $1/(1+C(\lambda)G(\lambda))$ between the disturbance and the error or, equivalently, the ℓ^1 norm of the impulse response.

We suppose that the plant is given as the quotient of two polynomials $p(\lambda)$ and $q(\lambda)$ without common zeros and with no zeros on the unit circle, $G(\lambda) = p(\lambda)/q(\lambda)$.

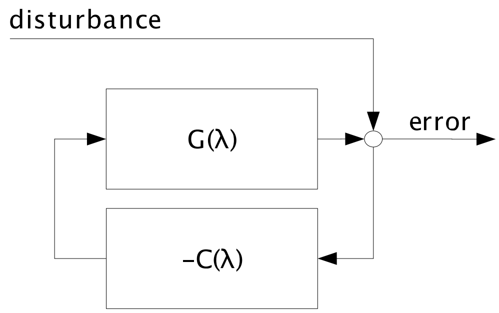

FIG. 12.1. *Standard feedback control configuration.*

The Youla–Kučera parametrization of all stabilizing controllers is

$$C(\lambda) = \frac{v(\lambda) + q(\lambda)x(\lambda)}{w(\lambda) - p(\lambda)x(\lambda)}, \tag{12.1}$$

where $v(\lambda), w(\lambda)$ are polynomials determined by $G(\lambda)$ and $x(\lambda)$ is a function we can freely choose in the Wiener algebra. The entire procedure can be done in four steps.

Step 1. Find stable-unstable factorizations $p(\lambda) = p_s(\lambda)p_u(\lambda)$ and $q(\lambda) = q_s(\lambda)q_u(\lambda)$. Here the indices s and u label polynomials with all zeros inside and outside the unit circle, respectively. Efficient algorithms for stable-unstable factorization are known (see, e.g., [3] and the references cited therein). In particular, reliable FFT-based algorithms are available from [2], [14].

Step 2. Find polynomials $x_0(\lambda)$ and $y_0(\lambda)$ satisfying the Diophantine equation $q(\lambda)x_0(\lambda) + p(\lambda)y_0(\lambda) = 1$. This problem can be conveniently solved using the polynomial toolbox [20].

Step 3. The polynomials $v(\lambda), w(\lambda)$ in (12.1) are

$$v(\lambda) = q_u(\lambda)p_u(\lambda)y_0(\lambda), \quad w(\lambda) = q_u(\lambda)p_u(\lambda)x_0(\lambda).$$

Step 4. Inserting the result of Step 3 in (12.1) we obtain

$$\frac{1}{1+CG} = \frac{1}{1 + \dfrac{q_u p_u y_0 + qx}{q_u p_u y x_0 - px}\,\dfrac{p}{q}} = \frac{q q_u p_u x_0 - qpx}{q_u p_u (q x_0 + p y_0)},$$

which equals $qx_0 - q_s p_s x$ by virtue of Step 2. Thus, the final task is to minimize $\|y(\lambda)\|_W = \|q(\lambda)x_0(\lambda) - q_s(\lambda)p_s(\lambda)x(\lambda)\|_W$ or, in terms of the coefficient sequences, to minimize $\|y\|_1 = \|b - T(a_+)x\|_1$, where $b \in \ell^1$ is the coefficient sequence of $q(\lambda)x_0(\lambda)$ and $a_+(\lambda) = q_s(\lambda)p_s(\lambda)$. This problem can be solved using the algorithm of section 7. The desired optimal controller is given by (12.1) with $v(\lambda), w(\lambda)$ from Step 3 and $x(\lambda)$ from Step 4.

To have a numerical example, let

$$G(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{-45\lambda - 132\lambda^2 + 9\lambda^3}{-20 - 48\lambda + 5\lambda^2}.$$

The above procedure yields the Youla–Kučera parameter $x(\lambda) = 0.1321 - 0.0052\lambda$, the sensitivity function $y(\lambda) = 1.0000 - 12.5000\lambda - 37.5000\lambda^2$, and the optimal controller

$$C(\lambda) = \frac{-41.6667 + 4.1667\lambda}{-7.5000 + 113.0000\lambda - 7.5000\lambda^2}.$$

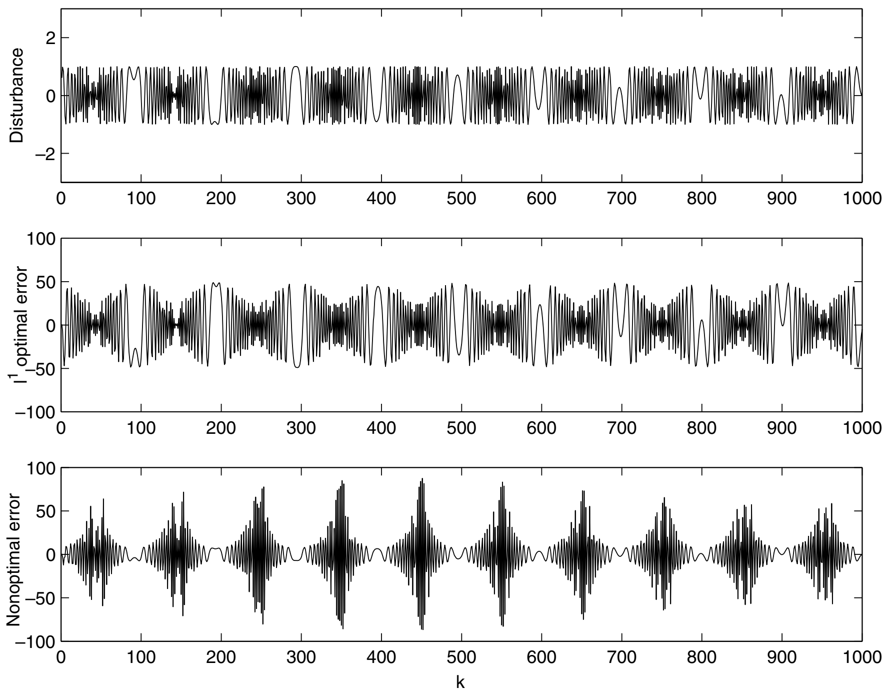

FIG. 12.2. *Simulation of a disturbance rejection with ℓ^1-optimal and nonoptimal controllers.*

A simulation result is shown in Figure 12.2. The horizontal axis represents the discrete time k. The disturbance is only known to be bounded in magnitude. The response of the closed-loop system to a disturbance bounded in magnitude by 1 is compared for the ℓ^1-optimal controller computed above and some random stabilizing controller. Note that with more sophisticated (optimal) controllers like LQG, $\mathcal{H}_2$-, and $\mathcal{H}_\infty$-optimal controllers the difference will not necessarily be this striking, as the system norms in finite-dimensional spaces are equivalent and, loosely speaking, minimizing some norm causes all the other norms to be small as well.

13. Conclusions. This paper contains an analysis of the problem of finding the minimum distance to the range of a banded lower-triangular Toeplitz operator in ℓ^1. Necessary and sufficient conditions for the existence of a minimizing sequence in ℓ^1 are established. It is shown that the optimal error sequences is finitely nonzero. The application of this result in ℓ^1-optimal control is outlined. The paper presents an alternative formulation and a new approach to the well known problem, putting it into operator-theoretic framework and leading to more reliable numerical algorithms.

It is demonstrated by an example that in some situations the interpolation approach may lead to highly ill-conditioned linear systems while the method proposed does not suffer from this unpleasant circumstance. Another striking feature of the present approach is that the (stable) optimal Youla–Kučera parameter is returned as a direct outcome of the linear optimization, so that there is no need for any numerically tricky extraction of an optimal controller from the closed-loop transfer function. Within the interpolation framework, a solution to the truncated problem does not necessarily yield an achievable stable closed-loop transfer function. Hence the controller obtained from it need not stabilize the plant. These complications are more pronounced in the general MIMO case when only suboptimal solutions can be obtained.

The authors admit that the paper might seem unnecessarily tough to read, but the motivation for this high level of abstraction and rigorousness was to attract researchers from outside the field control theory, especially from operator theory. For instance, at this moment it is not known how to get an estimate of the length of the optimal error sequence (however, it is known that this length is data dependent [22]). Also, it is not known how to build a finite-dimensional approximation to the dual problem such that its optimal solutions are feasible for the original dual problem. This is a vital issue for the development of general primal-dual solvers. The extension of the analysis presented here to operators on vector-valued infinite sequences is very attractive in connection with practical applications, because it would enable control of systems with more than one control/disturbing input and more than one measured and/or regulated input. Some progress recently has been made by the authors, and preliminary results on the straightforward extension to rectangular block Toeplitz operators including analysis of existence, uniqueness, and convergence were presented in [16]. The present paper can be regarded as a building block for this extension.

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